The Split Density Model:
A unified Description of Polarization and Array Dynamics for Vertical Cavity Surface Emitting Lasers

Holger F. Hofmann and Ortwin Hess

Institute of Technical Physics, DLR

Pfaffenwaldring 38–40, D–70569 Stuttgart, Germany

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We present a generalized rate equation model for the array and polarization dynamics of general one or two dimensional arrays of Vertical Cavity Surface Emitting Lasers (VCSELs). It is demonstrated that our model includes both the previous theory for edge emitting laser arrays and the theory of polarization dynamics in quantum well VCSELs in a single unified description. The model is based on the physical assumption of separated carrier densitiy pools individually coupled to different light field modes. These modes, defined by the separate gain profile of each pool, interact through the coherent dynamics of the light field derived from Maxwell's equations and represented by the coefficients for index and loss guiding. The special case of two densities and two light field modes is solved and the implications of the results for large VCSEL arrays are discussed. Our analytic results show that typical solutions of the split density model range from phase locking to chaos, depending on the magnitude of the frequency coefficients. For weak coupling, the stable supermode is always the mode of highest frequency. This indicates that anti-phase locking is the only stable phase locking possible in arrays.

I. INTRODUCTION

If vertical cavity surface emitting lasers (VCSELs) are fabricated in the form of a densely packed two dimensional array as shown in Fig.1, their highly symmetric geometry around the axis of emission calls for an investigation of both polarization dynamics and laser array interactions. Previously these two aspects have been treated separately. Experimental investigations of VCSEL arrays [1–3] usually cite theories derived for the one dimensional case of side-by-side edge emitting laser arrays such as the theory introduced by Winful and Wang in 1988 [4]. On the other hand, a model for the description of polarization dynamics in individual VCSELs was introduced by San Miguel, Feng and Moloney in 1995 [5].

Both models are rate equation models for type B lasers. Adiabatic elimination of the dipole density is used to reduce the description of the electronic system to carrier densities. However, the similarity does not end here. In both models, the carrier density is split into separate pools, each of which couples to a different light field mode. The differences between the models arise only from the choice of parameters which describe the dynamics of the electromagnetic field and the diffusive exchange of carriers between the carrier density pools.

Since the physical assumptions of the models for polarization and array dynamics are so similar, we will show in the following that indeed a general model for the polarization and array dynamics of arbitrary VCSEL arrays can be formulated. The coefficients of this generalized split density model will then contain all the information necessary to describe any special case. Therefore, the choice of the coefficient, which describe the physical properties of the device studied, is critical and a good understanding of the microscopic physics from which they can be derived is necessary to distinguish between common and exotic cases.

In section II, the physical justifications for introducing split densities in both the polarization and the array dynamics is discussed. Parameters for carrier diffusion and scattering are approximated. In section III, we present the general formalism of the split density model. The coefficients derived from Maxwell's equations are introduced and discussed. In section IV, the special case of two densities and two light field modes is treated. Fixed points and limit cycles are analytically calculated. In section V, our analytic results of the two density model are applied to large arrays. A stability analysis is presented and conditions for chaotic dynamics are discussed. In section VI, we conclude by discussing the strengths and weaknesses of the split density model as the basis of a description for general semiconductor laser array dynamics.

II. THE PHYSICAL JUSTIFICATION OF SPLIT DENSITIES

A. Polarization dynamics: separation of carrier density pools by angular momentum

In quantum well VCSELs, the axis of symmetry of the quantum well structure is identical to the axis of laser light emission. Therefore, the conservation of angular momentum may be applied to describe the emission process. Photons with an angular momentum of +1 (right circular polarized light) may only be emitted by electron-hole pairs with a total angular momentum of +1 and photons with an angular momentum of -1 (left circular polarized light) may only be emitted by electron hole pairs with a total angular momentum of -1. If only the bands closest to the fundamental gap are considered, electrons have an angular momentum of $\pm 1/2$

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and holes have an angular momentum of $\pm 3/2$. Consequently, electrons with an angular momentum of $+1/2$ may only recombine with holes of -3/2 angular momentum, emitting left circular polarized light. Electrons with an angular momentum of -1/2 may only recombine with holes of $+3/2$ angular momentum, emitting right circular polarized light.

Due to the conservation of angular momentum, the carrier density is therefore split into a carrier density emitting only right circular polarized light and a carrier density emitting only left circular polarized light. This effect was first included in a rate equation model by San Miguel and coworkers in [5]. In that model, it is assumed that the two densities interact by spin flip scattering processes which give rise to a diffusive exchange of carriers between the two densities. Experimental investigations of quantum well structures indicate that this scattering rate is of the order of $10^{10}s^{-1}$, similar to the carrier recombination rate [6]. The assumption of split densities should therefore be well justified if the separation of bands in the quantum well structure is sufficient to exclude the participation of higher bands in the laser process.

B. Array dynamics: spatial separation of carrier density pools

In a typical semiconductor laser array, carriers are injected into spatially separated regions of the active medium. If the distance of separation is sufficiently large, gain guiding effects will give rise to spatially separated optical modes, each associated by the gain function with its own carrier density pool. The localized modes are then coupled by Maxwell's equations, since the evanescent parts of the neighbouring optical modes overlap.

The rate equation theory for laser arrays of this type was developed by Winful and Wang in 1988 [4]. However, in that early formulation, carrier diffusion is completely neglected. To check the justification of that assumption, we need to estimate the time it takes carriers to diffuse from one localized carrier density pool to a neighbouring one. Noting that the microscopically computed ambipolar diffusion constant of charge carriers in GaAs based lasers is typically about $1\mu m^2/ns = 10\ cm^2/s$ [7], we can estimate the diffusive carrier exchange rate between lasers separated by a few micrometers to be close to $10^{10}s^{-1}$, similar to the spin relaxation rates expected for the polarization dynamics. However, the carrier exchange rates depend on the square of the distance between the lasers, which implies that the split density model rapidly loses its validity as the separation between elements of the array is reduced. In order to investigate the effects of diffusion, the possibility of carrier exchange between the densities is included in the generalized split density model presented in III.

In this model, carrier diffusion is described in terms of a carrier exchange rate. Of course, this is just a crude approximation of the real physical process of continuous diffusion which is quite different from the spin flip scattering involved in the carrier exchange between carrier density pools of different angular momentum. However, as long as the light field is only sensitive to the average carrier density within one laser of the array and higher transversal modes are neglected, the overall effect of diffusion can be summarized in a single carrier exchange rate.

We cannot but note that some of the coupled mode theories frequently cited use the assumption of homogeonous and stationary gain, ignoring the dynamical effects of spatially resolved carrier densities altogether [1,3,8–10]. In type B lasers, that assumption is only valid at the fixed points. As a consequence, those theories can not be used as starting points for an analysis of the stability and the dynamics of laser arrays.

III. GENERAL FORMALISM FOR THE DESCRIPTION OF VCSEL ARRAYS

A. Rate equations for the carrier densities: injection, non-radiative recombination and diffusion

The carrier densities of a VCSEL array can be separated into densities D_i associated with a position in the array and either left or right circular polarization. In a two dimensional VCSEL array, i represents the coordinates $(r, c, \pm)$ for row number r, column number c and plus for right circular polarization, minus for left circular polarization. When no electromagnetic field is present, the carrier dynamics including the effects of injection, non-radiative recombination and diffusion are

$$\frac{d}{dt}D_i = \mu_i - \gamma_i D_i - \sum_j T_{ij}(D_i - D_j), \qquad (1)$$

where μ_i is the effective local injection current, γ_i the local non-radiative recombination rate and $T_{ij} = T_{ji}$ is the rate of diffusive carrier exchange between the densities D_i and D_j. Note that the model assumes transparency at $\mu = 0$. When comparing the predictions of the model with experiment, it must be taken into account that μ represents the additional injection current after transparency has been reached. The real injection current is μ plus a device dependent constant. The non-radiative recombination rate and the carrier diffusion can conveniently be combined into a single relaxation matrix of the densities by defining

$$\Gamma_{ij} = \delta_{ij}(\gamma_i + \sum_k T_{ik}) - T_{ij}. \qquad (2)$$

Equation 2 describes the diffusive relaxation to an equilibrium of injection and recombination of electron-hole pairs. This constitutes the complete carrier dynamics at zero field. Typical relaxation rates are around $10^{10}s^{-1}$, as was discussed in II.

B. Maxwell's equations of the electromagnetic field: gain, loss and coherent propagation

The electromagnetic field inside the optical cavity of the laser array can be described by the complex field amplitudes of localized modes E_i of either left or right circular polarization. Each mode is associated with one density pool. In a two dimensional array, i represents the same coordinates $(r, c, \pm)$ given above. This is an essential feature of the split density model. The split densities correspond to a set of modes for the electromagnetic field which is different from the Eigenmodes of Maxwell's equations. Instead, these modes are Eigenmodes of the gain. They are the natural modes for the description of relaxation oscillations and limit cycles as well as the stability of fixed points. We will therefore chose this set of Eigenmodes and not the supermodes to describe the laser array. We note that we limit our theory to the fundamental mode of each laser in the array and disregard possible effects of spatial hole-burning within each laser. A generalization to include higher modes is possible, but would make it more difficult to distinguish array effects from the internal dynamics of individual lasers.

The dynamics of the electromagnetic field, including gain, loss and coherent propagation are given by Maxwell's equations, which are linear wave equations. In a general mode representation, they may be written as

$$\frac{d}{dt}E_i = w_i D_i (1 - i\alpha)E_i - \sum_j (\kappa_{ij} + i\omega_{ij})E_j. \qquad (3)$$

The carrier density dependent gain is given by $w_i D_i$. The rate of spontaneous emission into the laser mode w_i is typically around $10^6 s^{-1}$. Since most arrays will have very similar values for the different w_i, the average and the deviations from this average may be seperated into $w_i = \bar{w}(1 + f_i)$. In this context, f_i is the measure of the relative fluctuation in the coupling between density pool i and mode i.

The alpha factor α describes band structure effects which shift the gain profile to higher frequencies as the total density is increased. Typical values range from 2 to 6. Effectively, this shift of the gain profile introduces a carrier density dependent index of refraction.

κ_{ij} describes the losses through the mirrors. In most cases, the coefficients coupling different modes are very small and most of the losses are given by $\bar{\kappa}\delta_{ij}$ with $\bar{\kappa}$ around $10^{12} s^{-1}$. Small irregularities and birefringence in the mirrors may cause non negligible loss guiding effects which can be represented by small corrections $\bar{\kappa}l_{ij}$.

The matrix ω_{ij} describes the unitary coherent dynamics of Maxwell's equations based on the index of refraction of the array structure. Neighbouring modes interact by their overlapping evanescent waves, while right and left circular polarization modes are coupled by stress induced birefringence commonly found in epitaxially grown semiconductor heterostructures [15]. Since the magnitude of these effects is highly dependent on device archi-

tecture, it is difficult to estimate. However, their order of magnitude should be roughly between $10^{10}s^{-1}$ and $10^{12}s^{-1}$ to have a significant effect on the dynamics. For a given architecture, numerical simulations may be used to determine the effective coupling constant.

For stationary densities D_i, the dynamics of the electromagnetic field is strictly linear. It is therefore possible to find orthogonal Eigenmodes of the dynamics including the cooperative effects of gain guiding ($w_i D_i$), loss guiding (κ_{ij}) and index guiding (ω_{ij}). These modes are the supermodes of the coupled mode theories. Note that Eigenmodes of different density distributions D_i need not be orthogonal. Since semiconductor lasers are type B lasers, the dynamics of the D_i cannot be neglected and the modes defined by stationary D_i are next to useless for the investigation of dynamical effects and stability analysis.

C. Combined dynamics of the carrier density and the electromagnetic field

Now, the total dynamics of carrier density and electromagnetic field can be constructed by adding only the term of induced emission, i.e. $-2w_i E_i^* E_i D_i$, to the carrier density dynamics. The resulting equations of motion for N split densities and N complex field amplitudes representing, for example, an array of N/2 VCSELs with two polarizations each are

$$\frac{d}{dt}D_i = \mu_i - \sum_j \Gamma_{ij} D_j - 2\bar{w}(1 + f_i)E_i^* E_i D_i \qquad (4a)$$

$$\frac{d}{dt}E_i = (\bar{w}(1 + f_i)D_i(1 - i\alpha) - \bar{\kappa})E_i$$
$$- \sum_j (\bar{\kappa}l_{ij} + i\omega_{ij})E_j. \qquad (4b)$$

These two lines of equations already contain a large number of different physical effects. The localized modes in the array are not only coupled by overlapping evanescent waves (ω_{ij}) but also by carrier diffusion (Γ_{ij}) and by variations in the losses (l_{ij}). The injection current may be varied (μ_i) and the coupling of the active medium to the laser mode may also be different for each laser in the array (f_i).

In the following, we shall focus our attention on arrays pumped homogeneously ($\mu_i = \mu$) and with negligible variations in the gain and loss coefficients ($f_i = l_{ij} = 0$), resulting in

$$\frac{d}{dt}D_i = \mu - \sum_j \Gamma_{ij} D_j - 2\bar{w}E_i^* E_i D_i \qquad (5a)$$

$$\frac{d}{dt}E_i = (\bar{w}D_i(1 - i\alpha) - \bar{\kappa})E_i - \sum_j i\omega_{ij}E_j. \qquad (5b)$$

This equation describes the counteractive effects of coherent field dynamics represented by ω_{ij} and carrier diffusion represented by Γ_{ij}.

For the case of VCSEL polarization, the joint effects of coherent coupling and non-negligible gain-loss anisotropy, $l_{ij} \neq 0$, has been investigated by the authors in a separate paper [11].

IV. THE TWO DENSITY MODEL

A. Formulation of the rate equations

To understand how the effects of coherent field dynamics and carrier diffusion influence the stability of phase locking and the transition to chaos, we will now start our discussion with an investigation of the most simple case of two carrier density pools and two light field modes. This case describes either the polarization dynamics of a single VCSEL or an array of two edge emitting lasers. In these contexts, the equations of the two density model have been studied previously by Martin-Regalado (polarization) [12] and by Winful and Wang (array) [4] on the basis of numerical simulations. In the following we will review the results of the two density model and analytically derive general properties of array and polarization dynamics.

The equations for a symmetrical system of two carrier densitiy pools are defined by the parameters $\bar{w}$, $\bar{\kappa}$, Γ_{11}, Γ_{12}, and ω_{12}. It is useful to keep in mind that Γ_{11} is the sum of the carrier diffusion rate Γ_{12} and the non-radiative recombination rate γ. They read

$$\frac{d}{dt}D_1 = \mu - \Gamma_{11}D_1 + \Gamma_{12}D_2 - 2\bar{w}E_1^*E_1D_1 \qquad (6a)$$

$$\frac{d}{dt}D_2 = \mu - \Gamma_{11}D_2 + \Gamma_{12}D_1 - 2\bar{w}E_2^*E_2D_2 \qquad (6b)$$

$$\frac{d}{dt}E_1 = (\bar{w}D_1(1-i\alpha) - \bar{\kappa})E_1 - i\omega_{12}E_2 \qquad (6c)$$

$$\frac{d}{dt}E_2 = (\bar{w}D_2(1-i\alpha) - \bar{\kappa})E_2 - i\omega_{12}E_1. \qquad (6d)$$

All rates are given by positive and real values. This is not a natural assumption for ω_{12}, since a phase transformation in one of the two modes would generally also change the phase factor of ω_{12}. A real and positive ω_{12} implies that the $E_1 + E_2$ mode has a lower frequency than the $E_1 - E_2$ mode for $D_1 = D_2$ (no gain effects on the mode structure). In an array, this is a natural choice. If the modes describe orthogonal polarizations, however, the phase factor of ω_{12} is an important definition of the phase relation between the modes.

B. Stability analysis

If the two subsystems were not coupled by ω_{12}, they could be described as two identical lasers operating at the same injection current. Therefore, $D_1 = D_2$ and $E_1^*E_1 = E_2^*E_2$ is stationary. The introduction of ω_{12}

removes the arbitrary choice of phase and defines the supermodes with $E_1 = -E_2$ or $E_1 = E_2$ as the only possible stationary solutions for the light field.

The exact stationary solutions are given by $D_1 = D_2 = \bar{\kappa}/\bar{w}$ and $E_1^*E_1 = E_2^*E_2 = \mu/2\bar{\kappa} - \gamma/2\bar{w}$. To analyse the stability of the two possible supermodes, small fluctuations around these stable values are defined:

$$\delta D = D_1 + D_2 - \frac{2\bar{\kappa}}{\bar{w}} \qquad (7a)$$

$$\delta n = E_1^*E_1 + E_2^*E_2 - \frac{\mu}{\bar{\kappa}} - \frac{\gamma}{\bar{w}} \qquad (7b)$$

$$d = D_1 - D_2 \qquad (7c)$$

$$s_1 = E_1^*E_1 - E_2^*E_2 \qquad (7d)$$

$$s_2 = i(E_1^*E_2 - E_2^*E_1). \qquad (7e)$$

Note that in the case of polarization dynamics, s_1 and s_2 are two of the three Stokes parameters. The third Stokes parameter is defined as $s_3 = E_1^*E_2 + E_2^*E_1$ and is equal to plus or minus the total intensity for the two stationary solutions.

Using these parameters, the linearized dynamics around the stationary point of the antisymmetric supermode with $E_1 = -E_2$ and $s_3 = -n_s$ is

$$\frac{d}{dt}\begin{pmatrix} \delta D \\ \delta n \\ d \\ s_1 \\ s_2 \end{pmatrix} =$$

$$\begin{pmatrix} -\gamma - \bar{w}n_s & -2\bar{\kappa} & 0 & 0 & 0 \\ \bar{w}n_s & 0 & 0 & 0 & 0 \\ 0 & 0 & -\gamma - \bar{w}n_s - 2\Gamma_{12} & -2\bar{\kappa} & 0 \\ 0 & 0 & \bar{w}n_s & 0 & \omega_{12} \\ 0 & 0 & \alpha\bar{w}n_s & -\omega_{12} & 0 \end{pmatrix} \begin{pmatrix} \delta D \\ \delta n \\ d \\ s_1 \\ s_2 \end{pmatrix},$$

$$(8)$$

where $n_s = E_1^*E_1 + E_2^*E_2$ is the total intensity of the stationary solution. To obtain the stability matrix for the symmetric supermode with $E_1 = E_2$ and $s_3 = +n_s$, one only has to change the sign of ω_{12}.

The stability matrix can be separated into two blocks. One two by two block describes the relaxation oscillations of the total intensity. The remaining three by three block contains all the relevant information on the interaction between the subsystems. For weak coupling, s_1 and d show relaxation oscillations damped by diffusion, while s_2 will be stabilized or destabilized exponentially by its interaction with d via α.

The condition for the stability of a solution is, that the real parts of all Eigenvectors of the matrix given above are negative. For the symmetric supermode $E_1 + E_2$, which is the lowest frequency mode, this condition may be found by searching for a parameter set with one Eigenvalue equal to 0. This is done by setting the determinant of the stability matrix to 0. The result of this analysis is

$$\omega_{12} > 2\bar{\kappa}\alpha\frac{\bar{w}n_s}{\gamma + \bar{w}n_s + 2\Gamma_{12}}. \qquad (9)$$

Since we must assume that the diffusion rate Γ_{12} is not much larger than the non-radiative recombination rate γ for the split density model to be valid and since the rate of induced emission $\bar{w}n_s$ cannot be much larger than γ for reasonable injection currents, this condition can only be fulfilled if ω_{12} is as large as the loss rate $\bar{\kappa}$. This would require frequency separations of THz between the supermodes, which is unlikely if any reasonable gain, index or loss guiding exists to separate the modes. Since such guiding effects are absolutely necessary to define the array, the low frequency solution is only stable if the description of the device as an array of coupled lasers breaks down. It is therefore reasonable to assume that in a laser array guiding effects reduce ω_{12} to values below $\bar{\kappa}$. Consequently only the highest frequency supermode can be stable in a two mode array. If a different supermode is stabilized, this is most likely not an array effect, but instead represents higher mode effects typical of a broad area laser device [13,14]. In the case of polarization dynamics in VCSELs, a birefringence effect of THz is unrealistically high. Experiments indicate that ω_{12} is more likely to be of the same order of magnitude as γ [15]. Therefore, the stable polarization corresponds always to the one with the highest frequency. Of course, the stability condition for the low frequency mode can still be fulfilled very close to threshold. This gives rise to a bistability which was reported in [12]. However, according to our estimates, this bistability should only occur extremely close to threshhold, where noise effects dominate the laser dynamics. This bistability is therefore not likely to be of any practical relevance.

The condition for stablility of the anti-symmetric supermode $E_1 - E_2$ is found by searching for parameters permitting the existence of purely imaginary Eigenvectors. This indicates the point at which the relaxation oscillations of s_1 and d become undamped. The condition for damped relaxation oscillations is found to be

$$\alpha\omega_{12} < \gamma + \bar{w}n_s + 2\Gamma_{12}. \tag{10}$$

This condition is always fulfilled if the damping effect of the diffusion rate $2\Gamma_{12}$ exceeds the undamping effect of the coherent coupling $\alpha\omega_{12}$. Effectively, a moderate amount of diffusion stabilizes phase locking and prevents the transition to limit cycles and chaos. If $\alpha\omega_{12}$ exceeds $\gamma + 2\Gamma_{12}$, a minimal intensity is necessary to stabilize phase locking. Possibly, this intensity is unrealistically high, so that injection currents far above threshold would be necessary to stabilize the dynamics. Otherwise, a transition from stability to a limit cycle will be observable as the injection current is lowered.

C. Dynamics close to the fixed point: damping and undamping of relaxation oscillations

The analytic results of the stability analysis show that both semiconductor laser arrays and birefringent VCSELs tend to emit light in the mode of highest frequency.

In order to provide a foundation for the discussion of limit cycles and for the extrapolation of the results to larger arrays, we will now take a closer look at the dynamics of fluctuations around the fixed point. This is done by finding the Eigenvectors and Eigenvalues of the stability matrix. Although, in principle, approximations are not necessary to diagonalize the two by two and three by three blocks of the stability matrix it helps the interpretation of the physics if we make use of the fact that for semiconductor lasers, the cavity loss rate $\bar{\kappa}$ is typically two orders of magnitude larger than the non-radiative recombination rate γ. As a result, the fluctuations of semiconductor lasers are dominated by fast relaxation oscillations of the frequency $\nu = (2\bar{\kappa}\bar{w}n_s)^{1/2}$. The stability matrix can then be diagonalized by treating the damping term $\gamma + \bar{w}n_s + 2\Gamma_{12}$ and the coherent coupling ω_{12} as small perturbations of the relaxation oscillations. In lowest order, the Eigenvalues λ_i and the left and right Eigenvectors $\mathbf{a}_i$ and $\mathbf{b}_i$ are

$$\lambda_{1/2} = -\tfrac{1}{2}(\gamma + \bar{w}n_s) \pm i\nu$$

$$\mathbf{a}_{1/2} = \tfrac{1}{\sqrt{2}}\left(\ \mp i\sqrt{\tfrac{\bar{w}n_s}{2\bar{\kappa}}}\ \ 1\ \ 0\ \ 0\ \ 0\ \right)$$

$$\mathbf{b}_{1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} \pm i\sqrt{\tfrac{2\bar{\kappa}}{\bar{w}n_s}} \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11a}$$

$$\lambda_3 = -\alpha\omega_{12}$$

$$\mathbf{a}_3 = \begin{pmatrix} 0 & 0 & 0 & -\alpha & 1 \end{pmatrix}$$

$$\mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{11b}$$

$$\lambda_{4/5} = -\tfrac{1}{2}(\gamma + \bar{w}n_s + 2\Gamma_{12} - \alpha\omega_{12}) \pm i\nu$$

$$\mathbf{a}_{4/5} = \tfrac{1}{\sqrt{2}}\left(\ 0\ \ 0\ \ \mp i\sqrt{\tfrac{\bar{w}n_s}{2\bar{\kappa}}}\ \ 1\ \ 0\ \right)$$

$$\mathbf{b}_{4/5} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ \pm i\sqrt{\tfrac{2\bar{\kappa}}{\bar{w}n_s}} \\ 1 \\ \alpha \end{pmatrix}. \tag{11c}$$

In (11a) and (11c), solutions 1 and 4 refer to the upper sign and solutions 2 and 5 refer to the lower sign, respectively.

The solution 1/2 describes the linearized relaxation oscillations of the total intensity and carrier density, solution 3 describes the stabilization of anti-phase locking and solution 4/5 describes the relaxation oscillations of the difference in the intensities and carrier densities.

The solutions 3 and 4/5 describe the supermode stability we are interested in. We note that an important feature of the Eigenvectors is their non-orthogonality. This is caused by the α factor. The α factor converts density fluctuations into phase fluctuations. An example to illustrate this effect can be given by the relaxation dynamics of a fluctuation in the intensity difference s_1, described at time $t = 0$ by the vector $(0, 0, 0, 1, 0)$. As shown in Fig. 2, this intensity fluctuation induces a phase fluctuation proportional to α as it relaxes.

The relaxation rates of the solution are influenced by this correlation between phase fluctuations and intensity fluctuations. In solution 3, the α factor converts the time derivative of the intensity difference, $d/dt\ s_1 = \omega_{12}s_2$, into a time derivative of the phase, $\lambda_3 s_2 = \alpha\ d/dt\ s_1$, causing stability or instability, depending on the mode. In solution 4/5, the time derivative of the phase is likewise converted into a time derivative of the intensity difference, increasing or decreasing stability. If the phase difference is stabilized, the relaxation oscillations are destabilized and vice versa. Mathematically, this is necessary because the sum of all Eigenvalues λ_i must be equal to the trace of the stability matrix.

Since the α factor increases frequency with increasing carrier densities, the equations show that only the supermode with highest frequency is stabilized in the presence of relaxation oscillations. However, the stability analysis has shown that strong coupling may destabilize this solution as well. In the Eigenvalues of solution 4/5, this is expressed by the negative contribution of $\alpha\omega_{12}$.

To understand the undamping effect given by $\alpha\omega_{12}$, let us recall that s_1 and s_2 oscillate in phase, with $s_2 = \alpha s_1$. Consequently, the matrix element ω_{12} causes an undamping of this oscillation by adding a term equivalent to $+\alpha\omega_{12}s_1$ to the time derivative of s_1. Effectively, we thus encounter the surprising property that the coherent dynamics of the light field pumps intensity from the mode at low intensity to the mode at high intensity. This destabilizing effect is countered by the carrier density relaxation effects when the carrier density difference d is maximal and the light field intensity difference s_1 is zero. In this way, non-radiative (γ) and radiative ($\bar{w}n_s$) carrier recombination and diffusion ($2\Gamma_{12}$) stabilize the relaxation oscillations by damping d, while coherent coupling ($\alpha\omega_{12}$) destabilize them by undamping s_1.

D. Low amplitude limit cycle

If the relaxation oscillations are undamped, the result is a low amplitude limit cycle. For small amplitudes, this limit cycle still resembles relaxation oscillations with

$$d(t) = A_0\sqrt{\frac{2\bar{\kappa}}{\bar{w}n_s}}cos(\nu t) \tag{12a}$$

$$s_1(t) = A_0 sin(\nu t). \tag{12b}$$

The damping term is still linear, since the diffusion and the carrier recombination terms are linear over a wide range of carrier densities. The undamping term is more complicated. It results from the phase dynamics

$$tan(\phi) = i\frac{E_1^*E_2 - E_2^*E_1}{E_1^*E_2 + E_2^*E_1} = \frac{s_2}{s_3} \tag{13}$$

of the phase difference ϕ between E_1 and E_2. For small fluctuations, $s_3 \approx -n_s$ and $tan(\phi) \approx \phi$, so that $s_2 \approx -n_s\phi$. If non-linear terms are included, the phase dynamics of the limit cycle is given by

$$\phi(t) = -\frac{\alpha}{n_s}A_0 sin(\nu t). \tag{14}$$

Since the total intensity n_s, the 1st Stokes parameter s_1 and the ratio between the 2nd and 3rd Stokes parameters (i.e. $tan(\phi)$) are known, we can determine s_2 using $n_s^2 = s_1^2 + s_2^2 + s_3^2$:

$$s_2(t) = \sqrt{n_s^2 - s_1(t)^2}sin(\phi(t)). \tag{15}$$

The 3rd order correction to the linear dynamics of s_2 is then given by

$$s_2(t) = \alpha s_1(t)(1 - s_1(t)^2\frac{3 + \alpha^2}{6n_s^2} + \ldots). \tag{16}$$

For small s_1/n_s, the term of 3rd order in s_1 reduces and flattens the maxima of s_2. To determine the time averaged effect on the amplitude A_0, the contributions to the time derivative of s_1 in phase with the oscillations must be found. This is the Fourier component of $sin(\nu t)$. Since the Fourier component of $sin(\nu t)$ in $sin^3(\nu t)$ is $3/4$, the total time averaged damping is

$$\frac{1}{2}(\gamma + \bar{w}n_s + 2\Gamma_{12} - \alpha\omega_{12}) + \frac{3}{8}\alpha\omega_{12}\frac{A_0^2}{n_s^2}\frac{3 + \alpha^2}{6}. \tag{17}$$

Setting this term to 0, the amplitude of the limit cycle is found to be

$$A_0 = 2n_s\sqrt{\frac{2}{3 + \alpha^2}(1 - \frac{\gamma + \bar{w}n_s + 2\Gamma_{12}}{\alpha\omega_{12}})}. \tag{18}$$

This equation describes the transition from stability to a limit cycle possible for strong coupling and low diffusion. The intensity dependent contribution to the stability is the rate of induced emission, $\bar{w}n_s$.

The amplitude of the limit cycle which has been calculated numerically in the paper on laser arrays by Winful and Wang [4] can now be determined analytically using this formula. That limit cycle was calculated at an injection current of 1.1 times threshold, while stability is reached at 2 times threshold for the parameters used. Fig.3 shows the dependence of A_0 on the injection rate for this choice of parameters.

V. GENERALIZATION OF THE TWO DENSITY MODEL RESULTS TO LARGER ARRAYS

A. Effective next neighbour interaction

Our analysis shows that in the case of stability, two interacting lasers will emit light 180 degree out of phase. This result can also be applied to larger arrays. Usually, the coupling constants ω_{ij} only couple next neighbours. If all ω_{ij} are real and positive, the stable mode will be the one in which next neighbours emit light 180 degree out of phase. Note that in the case of a VCSEL array, the two circular polarizations must be considered as separate members of the array, coupled by birefringence. In this sense, VCSEL polarization adds a third dimension to the otherwise two dimensional array.

In most conventional geometries, it is very easy to find a single stable supermode in which all next neighbours emit exactly out of phase. For example, a square array of VCSELs will emit light of a single linear polarization in the highest frequency supermode. In the far field, this corresponds to four intensity maxima emitted into directions tilted towards the corners of the array. This result has been observed experimentally [1], providing evidence that the split density model is a valid description for realistic devices.

In unconventional geometries such as triangular arrays or spatially varying birefringence, there may be more than one stable supermode. The stationary solutions may be obtained by letting the field amplitudes E_i be equal to $+E_0$ or $-E_0$. Among these modes, promising candidates for stability are always the arrangements with the highest possible number of phase changes between next neighbours.

Once the stable modes have been identified using the rule that next neighbours try to achieve anti-phase locking, the stability of this mode with regard to phase fluctuations and relaxation oscillations may be investigated. Since we can assume that the fastest time scale is that of the relaxation oscillations, the two effects may be separated. The justification for this separation is the same as the one underlying the stability analysis of IV: The Eigenvectors and Eigenvalues of the stability matrix may be determined by treating ω_{ij} and the effects of damping and diffusion as small perturbations of the fast relaxation oscillations. The 3N dimensional system of equations for the carrier densities and the real and imaginary parts of the fields of N lasers can then be separated into a system with 2N dimensions describing relaxation oscillations and one of N dimensions describing phase coupling.

In the following, we limit the coupling terms to next neighbour interactions, allowing a complete stability analysis of large symmetric laser arrays.

B. Stability of the anti-phase locked supermode

First, we will investigate the N dimensional stability matrix which describes the phase coupling of the array. In the two density model, the phase fluctuation was described by $s_2 \approx -n_s \phi$, where the phase difference $\phi = \delta\phi_1 - \delta\phi_2$ represents a measure of the difference between the phase fluctuations in system 1 and system 2. If $\delta\phi_i$ denotes the phase fluctuation in system i, the two density stability matrix for phase fluctuations is

$$\frac{d}{dt}\begin{pmatrix} \delta\phi_1 \\ \delta\phi_2 \end{pmatrix} = \begin{pmatrix} -\frac{\alpha\omega_{12}}{2} & +\frac{\alpha\omega_{12}}{2} \\ +\frac{\alpha\omega_{12}}{2} & -\frac{\alpha\omega_{12}}{2} \end{pmatrix}\begin{pmatrix} \delta\phi_1 \\ \delta\phi_2 \end{pmatrix}. \qquad (19)$$

This result may be generalized to larger arrays by using (19) as next neighbour interaction of phase fluctuations in arbitrary arrays. The general equation for the dynamics of phase fluctuations is then given by

$$\frac{d}{dt}\delta\phi_i = \sum_j \frac{\alpha\omega_{ij}}{2}(\delta\phi_j - \delta\phi_i). \qquad (20)$$

This equation describes the linear relaxation of phase fluctuations in an array of arbitrary size and geometry. The stability matrix is defined by the $\alpha\omega_{ij}$. Note, that the relaxation has the properties of a diffusion such as the one described by T_{ij} for the carrier densities as introduced in III. This indicates that local phase changes diffuse to the neighbouring systems until the whole array is again phase locked. For an array with lattice constant a_0, the phase diffusion constant is approximately given by $a_0^2\alpha\omega_{ij}$. Thus, for micrometer scale arrays, typical phase diffusion constants are probably around $1\mu m^2/ns$. This is the same order of magnitude as the carrier diffusion constant, even though the physical properties defining each are quite independent of each other.

C. Anti-diffusion and the route to chaos

The remaining 2N variables of the stability matrix describe relaxation oscillations of the carrier densities and the light field intensities of the array.

In the two density model, the relaxation oscillation amplitude of both the total intensity and the intensity difference are damped by the non-radiative and the radiative recombination of carriers, i.e. $\gamma + \bar{w}n_s$. In addition, the oscillations of the intensity difference are damped by diffusion, $2\Gamma_{12}$ and undamped by the coherent interaction between the two systems, $\alpha\omega_{12}$. Again the coherent interaction acts as a diffusion process. However, this time the diffusion constant is negative: the coherent coupling gives rise to anti-diffusion, increasing the intensity difference between next neighbours.

If we denote the local amplitude of the relaxation oscillations in the subsystem i by the complex value A_i, the two density result may be generalized to arbitrary arrays

on the basis of the same principles applied in the previous subsection to the phase fluctuations. The dynamics of the A_i averaged over several relaxation oscillations is then given by

$$\frac{d}{dt}A_i = -\frac{1}{2}((\gamma_i + \bar{w}n_s)A_i + \sum_j (T_{ij} - \frac{\alpha\omega_{ij}}{2})(A_i - A_j)).$$

(21)

Note, that Γ_{ij} has been decomposed into the contributions from non-radiative carrier recombination γ_i and from diffusion T_{ij}, because they contribute to different types of effects. $\gamma_i + \bar{w}n_s$ causes an overall relaxation of the oscillations, while the diffusion and anti-diffusion terms $T_{ij} - \alpha\omega_{ij}/2$ are sensitive only to amplitude differences between nearest neighbours. For $\gamma_i = \gamma_0 = constant$, the relaxation effect stabilizes all forms of relaxation oscillations equally well. The diffusion stabilizes especially the relaxation oscillations with an amplitude which varies strongly between neighbouring systems. Consequently, these relaxation oscillations are also undamped most effectively when $T_{ij} < \alpha\omega_{ij}/2$. In a regular lattice, the least stable relaxation oscillation in the presence of such anti-diffusion is the one in which $A_i = -A_j$ for next neighbours. In a square lattice with constant coherent interaction ω_{nn} and constant diffusion T_{nn} between next neighbours, the stability condition for this oscillation is

$$\alpha\omega_{ij} < 2T_{ij} + \frac{1}{4}(\gamma_i A_i + \bar{w}n_s)$$

(22)

When $\alpha\omega_{ij}$ exceeds this limit, the relaxation oscillations become undamped and the near field intensity oscillates in a chessboard pattern. Since this implies that the intensities of all VCSELs emitting in the same phase oscillate in parallel, this oscillation should show up as a small intensity peak in the otherwise dark center of the far field pattern.

For stronger anti-diffusion, more and more relaxation oscillations become undamped and the amplitude of the oscillations increases. The chessboard pattern of the oscillations will then be washed out and non-linear effects shift the frequencies and add higher harmonics. For very strong coupling, the neighbouring VCSELs will switch on and off in an unpredictable chaotic fashion.

VI. PRACTICAL IMPLICATIONS AND OPEN QUESTIONS

The split density model is the most simple way of describing the array and polarization dynamics of VCSELs. It is also applicable to the array dynamics of conventional edge emitting arrays.

For the most interesting case of large symmetric arrays, the results of the split density model are strikingly simple and clear: if the separation between the lasers in the array does not break down because of rapid diffusion, next neighbours lock into anti-phase laser emission. Consequently, only the highest frequency supermode is stabilized.

For VCSEL arrays, this behavior has indeed been observed experimentally in [1–3]. In [1], this effect is explained as the result of higher losses between the VCSELs in the array. In the split density model, this corresponds to a coupling via the l_{ij} terms introduced in III. Such terms do indeed stabilize the highest order supermode to a certain degree. However, since the l_{ij} are proportional to the weak intensities, while the ω_{ij} terms are proportional to the fields between the VCSELs , this effect is likely to be much smaller than the coherent interaction. We therefore conclude that the explanation given in [1] for the experimentally observed anti-phase locking may be a misinterpretation caused by the neglect of carrier dynamics in many coupled mode theories [8–10]. Those theories certainly ignore the stabilizing effects of carrier dynamics and coherent interaction, which cannot be neglected in type B lasers.

Another aspect of the split density model, pioneered in the numerical work by Winful and Wang [4], is the existence of limit cycles and chaos in semiconductor laser arrays. We have calculated the stability limit including carrier diffusion effects and have shown that stability is not as unlikely as predicted in the original paper by Winful and Wang. Indeed, carrier diffusion may stabilize phase locking.

A careful analysis of the fluctuations in the center of a far field pattern such as the one observed in [1] could reveal whether the array is well stabilized by diffusion (weak fluctuations in the center), shows anti-diffusion (fluctuations have a maximum in the center) or is already oscillating in a weak limit cycle (coherent light is emitted into the center of the far field). Possibly, the correct range of parameters necessary for limit cycles and weak chaos could be realized experimentally by varying the distance between the VCSELs to change diffusion effects and by using different types of optical guiding. For larger separations, pure gain guiding may be sufficient, while loss guiding is better for very small arrays.

In the same spirit, VCSEL polarization may be investigated, using the quantum well width to change scattering rates and stress to induce birefringence. Here, too, the frequency spectrum of the polarization orthogonal to the laser mode reveals a lot about the mechanism of stabilization in the VCSEL and may actually include coherent emissions from a limit cycle. Possibly, some of the VCSELs classified as having only poor polarization stability in reality show a well stabilized high frequency limit cycle and it is the time average of this limit cycle which is observed in polarization measurements.

Of course, when interpreting experimental results, the underlying assumptions made in the split density model must be kept in mind. If the diffusion or the spin flip scattering is too strong, the split densities must again be considered as a single carrier density and the supermodes

become higher modes of a single laser with many modes. Consequently, the split density model can be used to describe the transition from split to unified density. This has been attempted for polarization dynamics in quantum well VCSELs by Martin-Regalado in [12]. However, that approach is only valid for a narrow range of parameters and it may be debatable if the limitation to the lowest quantum well bands implied in [5,12] is still justified in quantum wells with such a strong spin flip scattering.

If the injection current is increased, higher order modes of the individual lasers in the array may also participate in the dynamics. However, to simplify the presentation we have disregarded multi-transverse modes of an individual laser. When considering that case, one has to assume that the dynamics is not governed by array effects, but instead by phenomena typical of broad area laser dynamics. The transition between the two regimes can be seen in numerical simulations of gain guided edge emitting arrays [13,14]. In order to simulate the effect in the framework of the split density model, higher modes must be assigned to each density. However, this rapidly reduces the simplicity which makes the model so attractive.

In conclusion, we have shown that the split density model provides a simple description of semiconductor laser arrays and is equally useful for the description of VCSEL polarization. The results clearly show that the mode with the highest frequency is stabilized. This is in agreement with the experimental results reported in [1–3] and indicates that the handwaving explanation of the anti-phase locking given in [1] may indeed be wrong. We have seen that strong coupling may undamp the relaxation oscillations, leading through limit cycles to chaos. However, carrier difusion and both radiative and non-radiative carrier recombinations stabilize the anti-phase locked mode, which makes the stable operation of a VCSEL array more likely than the chaotic case.

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FIG. 1. Schematic geometry of a VCSEL array.

FIG. 2. Relaxation dynamics of the phase variable s_2 and the intensity difference s_1 induced by a fluctuation in the intensity difference s_1 at time $t = 0$. The choice of parameters is fully characterized by $\alpha = 3$, $\alpha\omega_{12} = \nu/15$ and $(\gamma + \bar{w}n_s + 2\Gamma_{12} - \alpha\omega_{12})/2 = \nu/5$. This example illustrates clearly, that fluctuations in the intensity difference s_1 induce strong fluctuations in the phase difference described by s_2.

FIG. 3. Dependence of the limit cycle amplitude A_0 on the carrier injection rate. The choice of parameters corresponds to that of [4] : $\alpha = 5$, $\Gamma_{12} = 0$ and $\alpha\omega_{12} = 2\gamma$. The injection rate in units of the threshold injection rate is equal to $1 + \bar{w}n_s/\gamma$. For this choice of parameters, the transition from the limit cycle to stabilization by induced emission is within the experimentally accessible range of injection current and would therefore be observable.

current contacts

dielectric mirrors

active layer